Conformal invariance and the regularised one-loop effective action

1988 J. Phys. A: Math. Gen. 21 L1131

(http://iopscience.iop.org/0305-4470/21/23/005)

J. Phys. A: Math. Gen. 21 (1988) L1131-L1134. Printed in the UK

LETTER TO THE EDITOR

Conformal invariance and the regularised one-loop effective action

Jeffrey Melmed

Institute of Field Physics, Department of Physics and Astronomy, University of North Carolina, Chapel Hill, NC 27599-3255, USA

Received 10 August 1988

Abstract. A unique family $\tilde{G}_{n-1}$ of conformal invariants, polynomial in the extrinsic curvature of a hypersurface embedded in an n-dimensional Riemannian manifold M, is constructed. In the quantum theory of a conformally coupled scalar field on a manifold M with boundary ∂M, the counterterm that regularises the one-loop effective action contains members of $\tilde{G}_{n-1}$ integrated over ∂M. The counterterms through $n=4$ are given explicitly and a derivation of the $n=4$ boundary contribution is given based on a flat-space result.

In the quantum theory of a massless conformally coupled scalar field on a general curved Riemannian manifold M, it is well known that one counterterm is required to regularise the divergent one-loop bare effective action (Birrell and Davies 1982). When the manifold is compact the standard counterterm is given by the volume integral of a particular combination of Riemann curvature invariants, which in the limit as $n \to \dim M$ (and only in the limit) becomes a global conformal invariant. This leads to the anomalous trace of the regularised effective stress-energy tensor. In the non-compact case the counterterm contains additional boundary integrals of invariants, depending on both the intrinsic and extrinsic curvature of the boundary surface. These, of course, do not affect the stress-energy tensor but do alter the value of the regularised effective action which is of considerable importance in obtaining the correct thermodynamics for closed systems.

In this letter we introduce $\tilde{G}_{n-1}$, a unique family of conformal invariants in n dimensions, constructed from the extrinsic curvature of an embedded $(n-1)$-dimensional hypersurface. We construct from the known flat-space boundary contributions (Kennedy 1978), the full counterterm for a conformally coupled scalar field on a curved 4-manifold. Furthermore, we show how the boundary contributions for the three- and four-dimensional theories depend on $\tilde{G}_{n-1}$. Dirichlet boundary conditions are assumed.

Given an n-dimensional manifold with boundary $(M, \partial M)$ and a metric g_{ab}, consider the scaling map or conformal transformation $\tilde{S}_n : g_{ab} \to \phi^4 g_{ab}$. When the manifold is a product $\Sigma_{n-1} \otimes \boldsymbol{R}$, and the metric is given by $g_{ab} = \gamma_{ab} + n_a n_b$, the map $\tilde{S}_n$ induces the map $\tilde{S}_{n-1}^{\text{induced}}$: $\sqrt{\gamma} \to \phi^{2(n-1)}\sqrt{\gamma}$; $P^{ab} \to \phi^{-6} P^{ab}$. In terms of the extrinsic curvature of the embedded hypersurface, P^{ab} is its trace-free part:

$$P^{ab} = K^{ab} - \frac{1}{(n-1)} K\gamma^{ab}.$$

It should be noted that $\tilde{S}^{\text{induced}}_{n-1} \neq \tilde{S}_{n-1}$ since the induced map includes the scaling of n_a while the other scales only the intrinsic metric γ_{ab}. (This is an important distinction, especially for the initial-value problem of the gravitational field where $\tilde{S}_{n-1}$ is relevant (York 1972).)

Under $\tilde{S}^{\text{induced}}_{n-1}$, a traced string of tensors $P^a{}_b$ containing s terms scales by a factor of ϕ^{-2s}. Consequently there exists a family $\tilde{G}_{n-1}$ of conformally invariant scalar densities (of weight 1), polynomial in $P^a{}_b$, with members $\sqrt{\gamma} g_{n-1}(V_i)$ defined by

$$g_{n-1}(V_i) \equiv \prod_{V_i} (P^{a_1}{}_{a_2} P^{a_2}{}_{a_3} \cdots P^{a_q}{}_{a_1}) \qquad \text{for } n \geqslant 2 \tag{1}$$

$$V_i = \{\text{integers } q > 1 | \textstyle\sum q = n-1\} \tag{2}$$

where i runs over the number of distinct such sets. For example, when $n=6$ there are two sets:

$$V_1 = \{5\} \to g_5(V_1) = \operatorname{Tr} P \cdot P \cdot P \cdot P \cdot P \tag{3a}$$

and

$$V_2 = \{2, 3\} \to g_5(V_2) = (\operatorname{Tr} P \cdot P) \cdot (\operatorname{Tr} P \cdot P \cdot P). \tag{3b}$$

Each of these invariants can be written as polynomials of $K^a{}_b$ and they are unique in that they are the only such polynomials of $K^a{}_b$ invariant under $\tilde{S}^{\text{induced}}_{n-1}$.

As we have noted already, the regularised quantum theory of a massless conformally coupled scalar field on a general curved Riemannian manifold requires one counterterm in the one-loop effective action W. The counterterm has a simple pole at $n = \dim M$ and is proportional to the nth coefficient, $C_{n/2}$, in an asymptotic expansion of the integrated heat kernel $K(\tau)$ of the operator $\square - \xi R$ where $\xi = (n-2)/4(n-1)$. The regularised action is given by

$$W_{\text{reg}} = W + \frac{(4\pi)^{-n/2}}{(n - \dim M)} C_{n/2} \tag{4}$$

where

$$W = \tfrac{1}{2} \int_0^\infty \frac{\mathrm{d}\tau}{\tau} K(\tau) \tag{5}$$

and the asymptotic behaviour is given by (Greiner 1971)

$$K(\tau) \sim (4\pi\tau)^{-n/2} \sum_{k=0}^{\infty} C_{k/2} \tau^{k/2} \qquad \text{as } \tau \downarrow 0. \tag{6}$$

The coefficients $C_{k/2}$ are given by

$$C_{k/2} = \int_M \mathrm{d}^n V \, a_{k/2}(x) + \int_{\partial M} \mathrm{d}^{n-1}\sigma \, b_{k/2}(x) \tag{7}$$

where the $a_{k/2}(x)$ are the so-called 'Hamidew' functions (Gibbons 1982) of the Riemann curvature of the manifold (which vanish for k odd), and the $b_{k/2}(x)$ are certain functions of the intrinsic and extrinsic curvature of the boundary. By zeta function techniques it can be shown that $C_{n/2}$ is a conformal invariant in n dimensions (Dowker and Kennedy 1978).

From explicit calculations of Kennedy *et al* (1980) we have that

$$C_0 = 0 \qquad n=0 \tag{8a}$$

$$C_{1/2} = 0 \qquad n=1 \tag{8b}$$

$$C_1 = -\tfrac{2}{3}\pi\chi_2 \qquad n=2 \tag{8c}$$

$$C_{3/2} = \frac{\pi^{1/2}}{96}\left(-8\pi\hat{\chi}_2 + 3\int_{\partial M} \mathrm{d}^2\sigma\, g_2(V)\right) \qquad n=3 \tag{8d}$$

where χ_n is the Euler–Poincaré characteristic (EPC) of an n-dimensional manifold with boundary (i.e. including boundary terms), $\hat{\chi}_{n-1}$ is the EPC of the boundary manifold alone and $g_2(V) = \mathrm{Tr}\, P\cdot P = \mathrm{Tr}\, K\cdot K - \frac{1}{2}(\mathrm{Tr}\, K)^2$. Clearly C_1 and $C_{3/2}$ are invariant under the maps $\tilde{S}_2$ and $\tilde{S}_3$, respectively. Using the conformal invariance of C_2 when $n=4$, we can generalise a flat-space calculation of C_2 given by Kennedy (1978):

$$C_2(\mathrm{flat}) = -\frac{1}{945}\int_{\partial M} \mathrm{d}^3\sigma\, f(K) \tag{9}$$

where

$$f(K) = 40\,\mathrm{Tr}\, K\cdot K\cdot K - 33(\mathrm{Tr}\, K)(\mathrm{Tr}\, K\cdot K) + 5(\mathrm{Tr}\, K)^3. \tag{10}$$

When the manifold is curved, the volume part of C_2 does not vanish since it comes from the integral of $a_2(x)$, which is a well known function of squared Riemann curvature terms. Therefore we may augment only the boundary integrand with terms that at least vanish when the Riemann tensor does, and which have dimensions of $(\mathrm{length})^{-3}$. There are apparently six linearly independent scalars that meet those requirements and we denote them collectively by

$$L_{\mathrm{R}}(\alpha_i) = \alpha_1 R\,\mathrm{Tr}\, K + \alpha_2 R_{ab}n^a n^b\,\mathrm{Tr}\, K + \alpha_3 R_{ab}K^{ab} + \alpha_4 R_{acbd}n^c n^d K^{ab} + \alpha_5 \pounds_n R + \alpha_6 \pounds_n (R_{ab}n^a n^b) \tag{11}$$

where $\pounds_n$ is the Lie derivative along the normal vector field of the boundary. The most general curved-space candidate is

$$C_2 = \frac{1}{360}\left(\int_M \mathrm{d}^4V[3H - G] - \int_{\partial M} \mathrm{d}^3\sigma[\tfrac{8}{21}f(K) + L_{\mathrm{R}}(\alpha_i)]\right) \tag{12}$$

with

$$H = R_{abcd}R^{abcd} - 2R_{ab}R^{ab} + \tfrac{1}{3}R^2 = (\mathrm{Weyl})^2 \tag{13}$$

and

$$G = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2. \tag{14}$$

The $\Box R$ term that usually appears in the volume (as part of $a_2(x)$) has been integrated by Gauss's law and absorbed into the α_5 term on the boundary. By redefining α_1, α_2, α_3 and α_4 and noting that

$$f(K) = 54 g_3(V) - 42\,\mathrm{det}_3(K) \tag{15}$$

where

$$g_3(V) = \mathrm{Tr}\, P\cdot P\cdot P = \mathrm{Tr}\, K\cdot K\cdot K - (\mathrm{Tr}\, K)(\mathrm{Tr}\, K\cdot K) + \tfrac{2}{9}(\mathrm{Tr}\, K)^3 \tag{16}$$

and

$$\det_3(K)=\frac{1}{3!}(2\,\mathrm{Tr}\,K\cdot K\cdot K-3(\mathrm{Tr}\,K)(\mathrm{Tr}\,K\cdot K)+(\mathrm{Tr}\,K)^3) \tag{17}$$

we can write

$$C_2=\tfrac{1}{360}\left(-32\pi^2\chi_4+3\int_M \mathrm{d}^4V\,H-\int_{\partial M}\mathrm{d}^3\sigma\,[\tfrac{144}{7}g_3(V)+L_{\mathrm{R}}(\alpha_i')]\right) \tag{18}$$

where χ_4 is the EPC of the 4-manifold with boundary. The first three parts of this are invariant under $\tilde{S}_4$, either locally or globally. Furthermore it is straightforward to show that the only set of coefficients $\{\alpha_i'\}$ which render the last term invariant (local or global) is the trival one $\{\alpha_i'=0\}$. Therefore we have the final result:

$$C_2=\tfrac{1}{360}\left(-32\pi^2\chi_4+3\int_M \mathrm{d}^4V\,H-\tfrac{144}{7}\int_{\partial M}\mathrm{d}^3\sigma\,g_3(V)\right). \tag{19}$$

The technique we have employed has necessarily caused us to miss the explicit dependence of C_2 on the coupling ξ. We have used the conformal coupling throughout so that the expression in (19) is the limit as $n\to 4$ of a general formula that depends on ξ. There may, in fact, be other terms which have coefficient $(\xi-\frac{1}{6})\propto(n-4)$.

Members of $\tilde{G}_{n-1}$ appear in the boundary integral for the counterterms beginning with $n=3$ where the first non-trivial one exists. When $n=3$ and $n=4$ there is a unique invariant comprising $\tilde{G}_2$ and $\tilde{G}_3$, respectively, and each of these forms part of the counterterm for the corresponding effective action. For $n\geqslant 5$ there are several members of $\tilde{G}_{n-1}$ but we have not given here their explicit contribution to the counterterms.

I thank all of my colleagues in Chapel Hill, especially B F Whiting and J W York for many stimulating conversations. This work was supported in part by the National Science Foundation grant no PHY-8407492.

References

Birrell N D and Davies P C W 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)
Dowker J S and Kennedy G 1978 *J. Phys. A: Math. Gen.* **11** 895
Gibbons G W 1982 *General Relativity, An Einstein Centenary Survey* ed S W Hawking and W Israel (Cambridge: Cambridge University Press) p 657
Greiner P 1971 *Arch. Rat. Mech. Anal.* **41** 163
Kennedy G 1978 *J. Phys. A: Math. Gen.* **11** L173
Kennedy G, Critchley R and Dowker J S 1980 *Ann. Phys., NY* **125** 346
York J W 1972 *Phys. Rev. Lett.* **28** 1082